Grade 6 Math Circles
November 7/8/9, 2023
Inequalities and Absolute Values

Introduction

By now you are likely comfortable with equalities. Math sentences like $2 + 4 = 6$ or $x = 10$ are equalities. The expression on the left side of the equals sign is equal to the expression on the right.

Today, we will explore inequalities. These math sentences involve expressions that are *not* equal to each other. Then we will explore the absolute values and how the absolute value affects inequalities.

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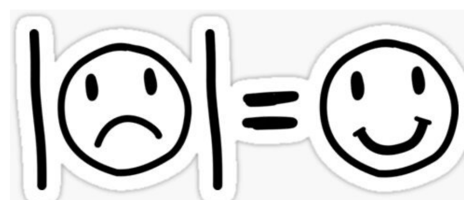

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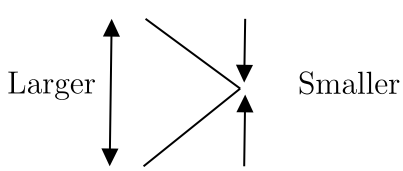
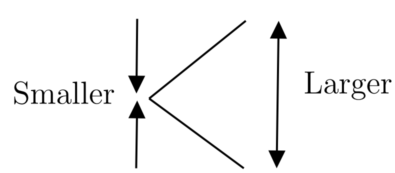
Inequalities

An **inequality** relates at least two expressions that are not necessarily equal. $<$ is the "less than" symbol and $>$ is the "greater than" symbol. These are called *strict* inequalities. To remember which symbol is which, look at the width between the ends of the symbol. The larger width opens to the larger number.



Greater Than



Less Than

Example 1

To say $4 < 5$, we would say "4 is less than 5." We can also flip the inequality to $5 > 4$ and we would say "5 is greater than 4."

Compound Inequalities

Compound inequalities use more than one inequality in a single sentence. For example, $0 < 4 < 9$ is a compound inequality and we would say "0 is less than 4 is less than 9."

Example 2

Annan is older than Ben. Ben is younger than Cathy. Cathy is younger than Annan.
Let a be Annan's age, b be Ben's age, and c be Cathy's age. Write a compound inequality that describes the ages above.

Solution:
Since Annan is older than Ben and Ben is younger than Cathy, Ben is the youngest of the three. We also know that Cathy is younger than Annan, so Cathy is between Ben and Annan. Therefore, the inequality is $b < c < a$.

TIP: You can also flip compound inequalities to suit the form you like. For example, if you are given $-4 > -7 > -7.5$ but you prefer when numbers are ordered from least to greatest, you can write the sentence as $-7.5 < -7 < -4$. These statements mean the same thing!

Exercise 1

Fill in the blanks with a <, >, or a number so each statement is true.

(a) 4 ____ -2

(b) $0 >$ ____

(c) ____ < -10

(d) 150 ____ -150

(e) -5 ____ 0

(f) ____ $>$ ____ > -13

Stop and Think

What are some applications of inequalities that we see in everyday life?

Sometimes we need to *solve* for a variable in an inequality. This requires some review of solving regular equations first.

Review of Solving Equations

If given equations with some variables like the ones below, you can solve for each variable. Let's practice. Solve the following fruit equations.

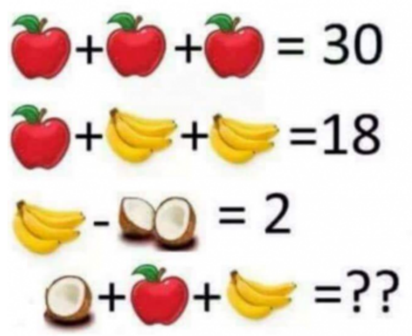


Image retrieved from The Problem Site

You might be able to do this without even writing anything, but what if we *had* to use algebra? The fruit problem above can be easily represented using math language by letting a represent one apple, b represent one banana, and c represent one coconut. Then you could solve for the variables.

Remember that when solving equations, you can:

- add and subtract **like** terms. Like terms are terms with the same variable. Eg. You can add bananas with bananas but can't add apples with bananas, $a + 4b + 4b = a + 8b$.
- perform operations like addition, subtraction, multiplcation, and division to both sides of the equation. Eg. If you are given $8b + 10 = 18$, then subtracing 10 from both sides gives $8b = 8$.
- replace a variable with the number it is equal to, if you have this information. Eg. If you know $a = 10$ and you want to solve for b given $a + 4b + 4b = 18$, then you can replace a with 10 to get $10 + 8b = 18$.

Example 3

Determine the sum of the fruit above using an algebraic solution by letting a represent one apple, b represent one banana in the bunch, and c represent one coconut. HINT: Pay close attention to the number of bananas and the number of coconuts in each photo.

Solution:

(1) Determine what a is equal to.

$$a + a + a = 30$$
$$3a = 30 \quad \text{(add like terms)}$$
$$a = 10 \quad \text{(divide both sides by 3)}$$

(2) Find what b is equal to.

$$a + 4b + 4b = 18$$
$$a + 8b = 18 \quad \text{(add like terms)}$$
$$10 + 8b = 18 \quad \text{(replace } a \text{ with 10 since } a = 10)$$
$$8b = 8 \quad \text{(subtract 10 from both sides)}$$
$$b = 1 \quad \text{(divide both sides by 8)}$$

(3) Determine what c is equal to.

$$4b - 2c = 2$$
$$4(1) - 2c = 2 \quad \text{(replace } b \text{ with 1 since } b = 1)$$
$$4 - 2c = 2$$
$$-2c = -2 \quad \text{(subtract 4 from both sides)}$$
$$c = 1 \quad \text{(divide both sides by } -2)$$

Therefore, $c + a + 3b = 1 + 10 + 3 = 14$.

Exercise 2

Solve the following equations.

(a) $2x - 7 = 23$

(b) $4y - y + 1 = 22$

(c) $4z + 1 = -5 + z$

Solve this system of equations.

(d) $6a - 1 = 29 + a$

AND

$2a - b = 16$

TIP: You can easily check if your answer is correct! If you substitute your answers back into the orignal equations, you should get a true equation. If you get something weird like $4 = 5$, then you know you made a mistake. Try this with your answers from Exercise 2!

Solving Inequalities

Inequalities can be solved using similar strategies as we used when solving equations.

- You can still add or subtract like terms and replace variables with their known value.
- You can add a term to or subtract a term from both sides and the inequality will still be valid.
- You can multiply or divide both sides by a **positive** term and the inequality will still be valid.
- If you multiply or divide both sides by a **negative** term you must **flip** the inequality sign.

Example 4

To see why you need to flip the sign when multiplying or dividing by a negative number, let's start with the true inequality, $2 < 4$.

If you multiply this by -1 without flipping the sign, you get $-2 < -4$. But this is FALSE!

Thus, flipping the sign to $-2 > -4$ maintains the correctness of the statement.

Try making up another example to double check that flipping the sign is necessary!

Example 5

Solve the following inequalities.

(a) $5 + x > -2$

(b) $y - 3y < 18$

Solution:

(a) Subtract 5 from both sides to obtain $x > -7$.

(b) Combine like terms to obtain $-2y < 18$.

Then divide both sides by -2 and flip the sign to obtain $y > -9$.

Exercise 3

Solve the following inequalities.

(a) $x + 2 > 11$

(b) $9 - 3a - a < 1$

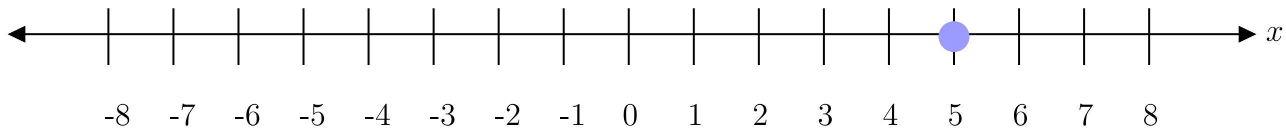
(c) $8 + 2b > 4b + 2$

Remember when we solved equations that each variable had only one number that satisfied the equation. Inequalities are usually satisfied by a range of numbers, not just one.

Representing Inequalities on a Number Line

Since inequalities often have a range of solutions, a number line is useful for visualizing this range.

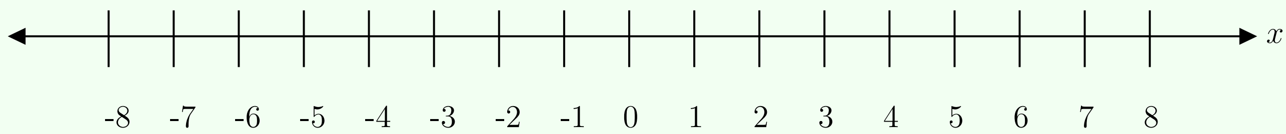
For equalities, the number line is not that exciting. It's simply a point at the number. Below is what $x = 5$ looks like.



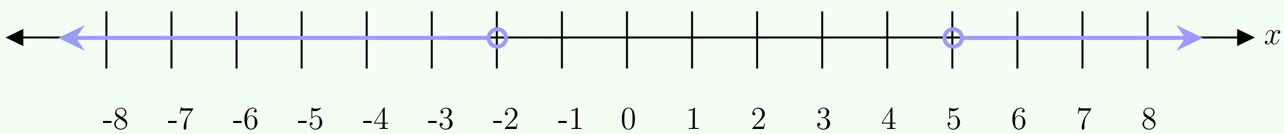
Number lines become more interesting when multiple inequalities are involved.

Example 6

Label $x > 5$ on a number line. Label $x < -2$ on the same number line.



Solution:



Notice that the circles for the strict inequalities are unshaded because -2 and 5 are excluded from the intervals.

To describe the values of x with words, we say x is less than -2 **or** x is greater than 5.

Note on "and" and "or"

The last example used **or** because $x < -2$ cannot be true at the same time as $x > 5$.

If the number line had $x > -2$, $x < 5$. Then we would use **and** because these can be true at the same time. We could write this as $-2 < x < 5$.

You will see more number lines and inequalities in the next section with the **Absolute Value**!

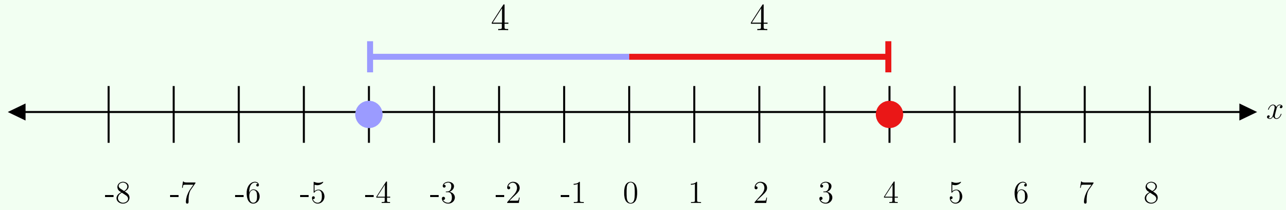
The Absolute Value

The absolute value of a number is its distance away from zero. Since distance cannot be negative, a number's absolute value is *always* positive. The symbol for the absolute value is $|\;|$. For example, $|-4|$ is "the absolute value of -4". Absolute values are also easier to visualize on a number line.

Example 7

Evaluate $|4|$ and $|-4|$ with the help of a number line.

Solution:



4 is four away from 0 and -4 is four away from 0. Therefore, $|4| = 4$ and $|-4| = 4$.

Simplifying Expressions with Absolute Values

The absolute value works kind of like brackets work. Thinking back to BEDMAS, brackets comes first so whenever there is an absolute value symbol, reduce anything inside of the absolute value and then evaluate the absolute value.

Exercise 4

Evaluate the following expressions. Ensure you follow the correct order of operations.

(a) $|5-3| + |3-5|$ (b) $|6 \times (-4)| + |-16 \div 2|$ (c) $-|2 + (-7)|$

Stop and Think

Does the order of subtraction matter inside of the absolute value?

If a negative sign is *in front* of the absolute value, will the number always be negative?

Solving Equations with Absolute Values

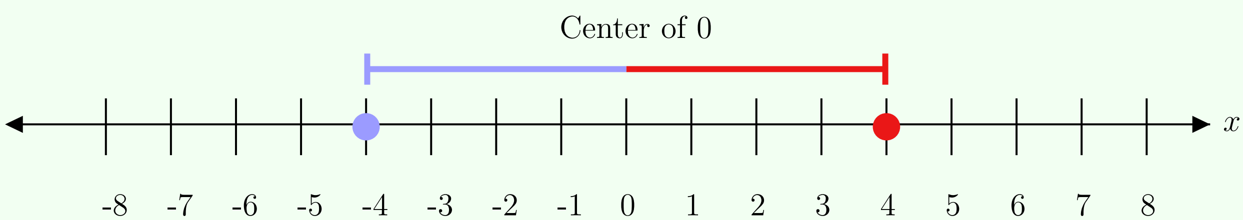
When solving equations with absolute values, we can use number lines to our advantage by "shifting" the number line to account for what is inside of the absolute value sign.

Example 8

(a) Determine the values of x such that $|x| = 4$.

Solution:

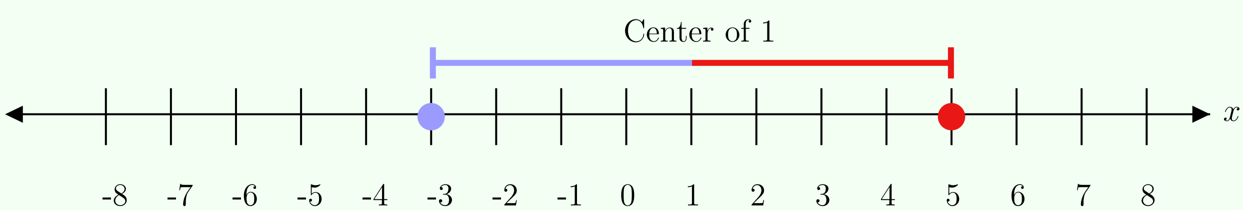
$|x| = 4$ asks us to find the numbers that are four away from 0. This means $x = 4$ **or** $x = -4$. Below is the number line that represents this.



(b) Determine the values of x such that $|x - 1| = 4$.

Solution:

$|x - 1| = 4$ asks us to find the numbers that are four away from 1 instead of 0 because the "-1" shifts the center to the right by 1. Below is the shifted number line.

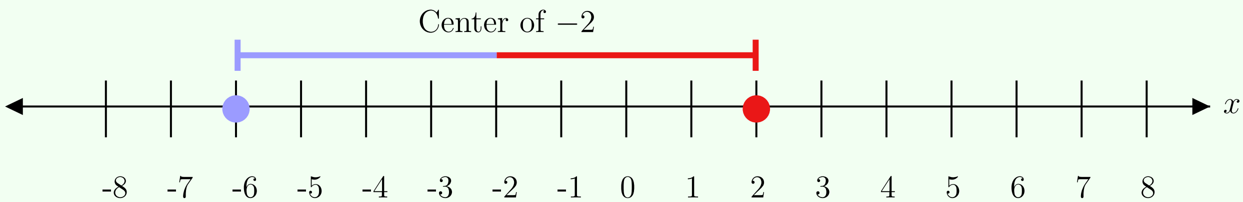


So, $x = 5$ **or** $x = -3$ because 5 is four away from 1 and -3 is four away from 1.

(c) Determine the values of x such that $|x+2| = 4$.

Solution:

$|x+2| = 4$ asks us to find the numbers that are four away from -2 instead of 0 because the "+2" shifts the center to the left by 2. Below is the shifted number line.



So, $x = 2$ **or** $x = -6$ because 2 is four away from -2 and -6 is four away from -2.

Notice how there are exactly *two* answers for x. This is different from similar equations without absolute values; they have exactly one solution.

Exercise 5

Determine the values of x such that the equations are true.

(a) $|x+5| = 9$

(b) $|x-3| = 7$

Solving Inequalities with Absolute Values

Now for the topic that combines all the ideas you have learned in this lesson. Remember how when solving inequalities, the answer is a range of numbers and when solving equations with absolute values, we got exactly two answers.

Stop and Think

How many answers do you think we will find when solving inequalities with absolute values?

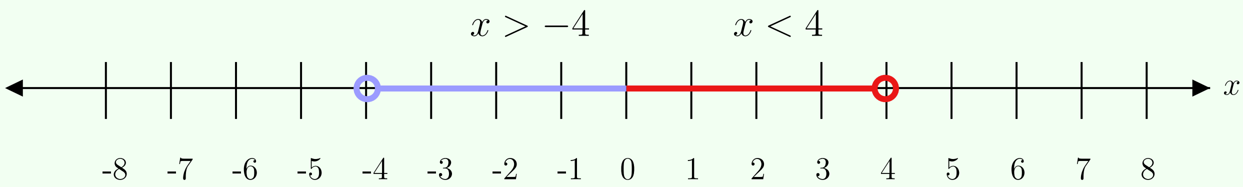
Example 9

(a) Determine the values of x such that $|x| < 4$ is true.

(b) Determine the values of x such that $|x| > 4$ is true.

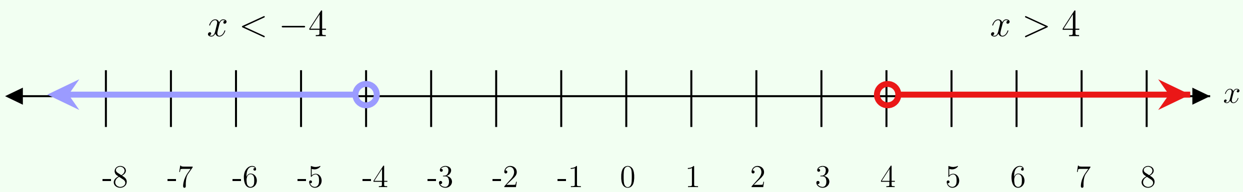
Solution:

(a) You can think of $|x| < 4$ as the distance between x and 0 is less than 4. Let's visualize this on a number line.



So, $|x| < 4$ when $x > -4$ **and** when $x < 4$. We can write this as $-4 < x < 4$.

(b) $|x| > 4$ means we need the distance from x to 0 to be greater than 4. We can draw this on the number line below.



This gives two *separate* inequalities. Since they are separate, we say $|x| > 4$ when $x < -4$ **or** $x > 4$.

Stop and Think

Do you think number lines are the only way to solve inequalities with absolute values or could we use another strategy instead?

The number line is a tool to help us visualize what the absolute value is, but of course it's not the *only* way to solve these inequalities! Problems in math than often be viewed from more than one angle. You can use algebra like we did with solving equations and inequalities. To do this, you need to consider two cases. First consider if x is greater than or equal to 0 and then if x is less than 0. This view of the absolute value leads to its *piecewise* definition, but this is a little complex at this level, so it is not included here.

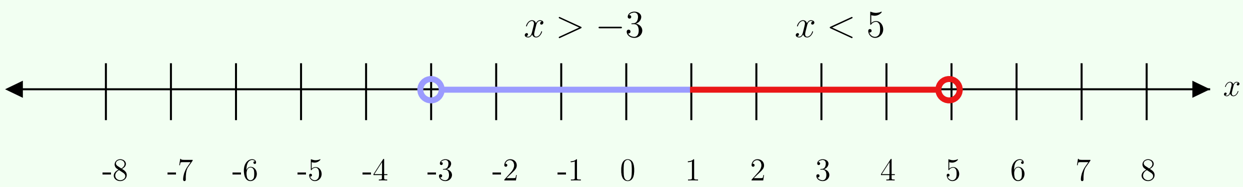
Example 10

(a) Determine the values of x such that $|x - 1| < 4$ is true.

(b) Determine the values of x such that $|x - 1| > 4$ is true.

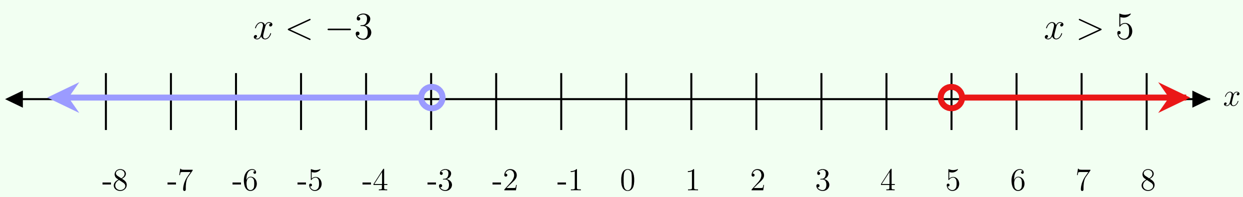
Solution:

(a) $|x - 1| < 4$ asks us to find the numbers that are less than four away from 1. Again, we can shift the number line to the right by 1.



The number line highlights the numbers that are less than four away from 1. We can see that $x > -3$ **and** $x < 5$. We can combine these to say $-3 < x < 5$.

(b) $|x - 1| > 4$ asks us to find the numbers that are greater than four away from 1. Again, we can shift the number line to the right by 1.



The number line highlights the numbers that are greater than four away from 1. We can see that $x < -3$ **or** $x > 5$. We cannot combine these because there is a gap between them.

Exercise 6

It's recommended that you use a number line to help with these questions.

(a) Determine all values of x such that $|x| < 8$.

(b) Determine all values of x such that $|x + 4| < 8$.

(c) Determine all values of x such that $|x - 2| > 3$.

Summary

Today you learned all about **inequalities and absolute values**! Inequalities involve expressions that are not equal to each other and the absolute value of a number is its distance away from 0.

This is just the beginning of these topics!

Inequalities appear in some useful theorems in mathematics like the Triangle Inequality, which states that the length of any two sides of a triangle is longer than its third side.

The absolute value can also be viewed from different definitions. The "distance from 0" definition makes the most sense visually, but there's also a piecewise definition which was hinted upon earlier in the lesson. You can also explore what the graph of the absolute values looks like. You will see all of these concepts later in your education!